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A STUDY ON THE THEORIES OF UNSTABLE CRACK EXTENSION FOR THE PREDICTION OF CRACK TRAJECTORIES

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Abstract—The paper deals with the prediction of the unstable crack trajectory from the stress field existing just before the onset of extension. A criterion termed as the criterion of zero shear stress has been introduced. Some finite element and experimental studies have been carried out. The crack trajectories based on the criteria of zero shear stress, minimum strain energy density (SED), and maximum tangential stress have been compared. All the three criteria appear to be useful for predicting the complete unstable trajectory of an internal crack in tensile panels. The SED criterion is unsuitable for the edge crack examples. The usefulness of the criteria of maximum tangential stress and zero shear stress for edge crack examples are shown. The latter is noted to have some computational advantages.

1. INTRODUCTION

The determination of resistance to fracture and the direction of initial crack extension have mostly provided the motivation for the development of the theories of fracture mechanics[1–3]. The study of crack trajectories has not received much attention. Some of the fracture criteria, e.g. the criteria of SED[4–6], maximum strain energy release rate[7], and maximum tangential stress[6] have been used for the determination of crack trajectories. Although the determination of crack trajectory may not be of any consequence for the prediction of strength, there are situations in metal working where study of the trajectory is of paramount importance. A preliminary study in this direction has been presented in reference[6]. In the present paper, the unstable crack trajectories obtained by applying the criteria of SED, maximum tangential stress and zero shear stress to some internal and edge crack problems have been compared. Theoretical analysis has been carried out using a finite element scheme. Some experimental studies connected with a typical mode of edge crack extension during shearing of bars are also reported.

2. THEORIES OF CRACK EXTENION

The earliest theory on the unstable crack extension is based on the global energy balance concept and is due to Griffith[1]. The criterion has been used for the prediction of the fracture load as well as the direction of initial crack extension. In the latter applications, it is considered that "the crack extends in the direction of maximum strain energy release rate[2]".

"Griffith has suggested that a crack would extend in a direction normal to the maximum tangential stress, and Erdogan and Sih[2], in applying this criterion, obtained an agreement with their experimental data"[4]. This criterion is now known as the criterion of maximum tangential stress. The criterion has also been used by Williams and Ewing[8] for the study of angled crack problems. With reference to Figure 1 therefore the direction of initial crack extention corresponds to the maximum of σ_θ.

Later Sih[3, 4] proposed the SED criterion according to which the crack extends in the direction of the minimum SED (Fig. 1) and the critical intensity of this potential field governs the onset of crack extension.

Hellen[7] studied a problem of stable fatigue crack extension using the Griffith's energy criterion in connection with the determination of stress intensity factors (SIFs) by the finite element method (FEM). The numerically predicted crack trajectory shows good agreement with his experimental observation. Kipp and Sih[4] considered that the unstable crack trajectory is predetermined by the stress field existing just before the onset of extension. They obtained the trajectory of an internal crack in a tensile panel as the locus of the point of minimum SED and verified the results experimentally. Similar theoretical and experimental studies on the unstable

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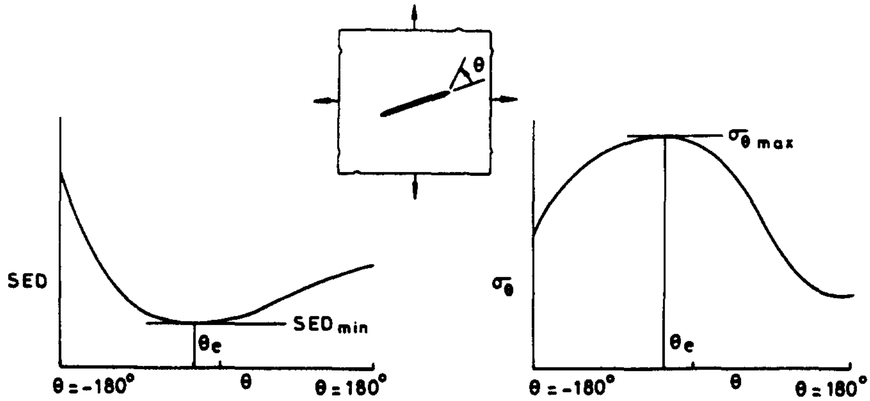


Fig. 1. Illustrations of the criteria of maximum tangential stress and minimum strain energy density.

extension have also been reported by Mau and Yang[5]. They applied the SED criterion considering both (i) the effect of redistribution of stress due to an infinitesimal crack extension and (ii) a fully unstable extension. The uneconomical step-by-step analysis adopted in Refs.[5,7] appears to have no particular advantage[5] and hence it is considered unsuitable for the present purpose. Both the tangential stress and the SED criterion have been used[6] in the global perspective to numerically predict unstable crack paths in some two dimensional problems extending to a length up to 5 times the initial crack. While applying the maximum tangential stress criterion it has been assumed that the unstable crack path is the locus of the point of maximum tangential stress. This preliminary numerical study indicated that the SED criterion may not be applicable to the problems of edge crack extensions.

The stress field around the crack tip B (Fig. 2) in the presence of two loading modes, I and II, is given[2] by

$$\sigma_\theta = \frac{1}{\sqrt{2\pi r}} \cos(\theta/2) \left[K_I \cos^2\frac{\theta}{2} - \frac{3}{2} K_{II} \sin\theta \right] \tag{1}$$

and

$$\tau_{r\theta} = \frac{1}{2\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) [K_I \sin\theta + K_{II}(3\cos\theta - 1)] \tag{2}$$

where K_I and K_{II} are the two SIFs. This stress field allows for the application of either $(\partial\sigma_\theta/\partial\theta) = 0$ or $\tau_{r\theta} = 0$ to determine the direction θ_c of initial crack extension (Fig. 1). Furthermore the maximum tangential stress at C (contiguous to B) is a principal stress.

The prediction of the entire unstable crack trajectory demands determination of the complete stress field including the crack tip region. For a certain class of problems the entire stress field

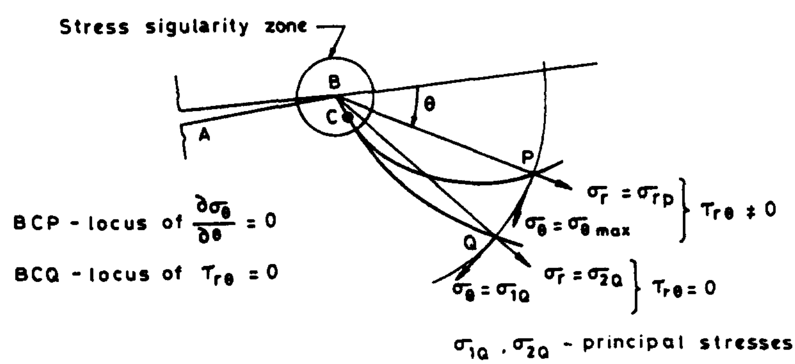


Fig. 2. Illustration of crack trajectories based on the criteria of maximum tangential stress and zero shear stress.

can be represented by a stress function ϕ of the William's[9] type:

$$\phi = \sum_{n=1}^{\infty} \phi_n \tag{3}$$

where

$$\phi_n = r^{(n/2)+1}\left[C_{2n-1}\left\{\sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\sin\left(\frac{n}{2}+1\right)\theta\right\}\right.$$
$$\left. + C_{2n}\left\{\cos\left(\frac{n}{2}-1\right)\theta - \cos\left(\frac{n}{2}+1\right)\theta\right\}\right] \tag{4}$$

C_{2n-1} and C_{2n} are the constants dependent on the geometry and boundary conditions.

For a general case, the stress function given by eqn(3) does not ensure $(\partial\sigma_\theta/\partial\theta) = 0$ and $\tau_{r\theta} = 0$ occuring at the same point. Therefore, at the point $\partial\sigma_\theta/\partial\theta = 0$ (P in Fig. 2), remote from the crack tip, the direction of the maximum tangential stress is not a principal direction and, consequently, the tangential stress is a "global" maximum but not a principal stress. Conversely, at the point $\tau_{r\theta} = 0$ (Q in Fig. 2) the radial direction is a principal direction and the tangential stress need not necessarily be the "global" maximum. Thus, at a given radius, $\partial\sigma_\theta/\partial\theta = 0$ gives the point of maximum tangential stress and $\tau_{r\theta} = 0$ gives the point at which the radial direction coincides with one of the principal directions.

Therefore, for a general case, outside the crack tip region, the two points, i.e. $\partial\sigma_\theta/\partial\theta = 0$ and $\tau_{r\theta} = 0$, are different. In this paper, it is examined whether the locus of the point of $\partial\sigma_\theta/\partial\theta = 0$ or $\tau_{r\theta} = 0$ represents the path of unstable extension of a crack.

The FEM[10] has developed significantly to deal with the elastic crack tip singularity and it has been extensively applied for the determination of SIF[11] corresponding to loading, geometry and boundary conditions which are in general beyond the scope of any analytical technique. The usefulness of the method for the prediction of unstable crack trajectories has been shown in Refs.[5 and 6] and it is further exploited in this paper.

3. NUMERICAL STUDIES

The first example pertains to the angled crack problem. The discretisation scheme is illustrated in Fig. 3. For a particular radius, the location of the minimum of SED, or maximum of σ_θ or zero of $\tau_{r\theta}$ is obtained by plotting (Fig. 4) all the Gauss point values along the circumference. These points, corresponding to the cases $\beta = 70, 60$ and $45°$ (Fig. 3), are shown in Fig. 5. The solid lines are due to Kipp and Sih[4].

The discretisation shown in Fig. 3 corresponds to $\beta = 60°$. The discretisation for the other two cases ($\beta = 70$ and $45°$) has been obtained by rotating the central core without altering the number of elements and the degrees of freedom.

The effect of size of the singularity elements has been studied by altering only r_0 (Fig. 3) and the results are presented in Fig. 5.

The edge crack extension has been studied in two cases shown in Fig. 6. The analysis is carried out with the assumptions[6] that the linear elastic fracture mechanics principles (LEFM) are valid and the state of stress is two-dimensional. The shearing load is assumed to be either uniformly or linearly distributed over the punch width l_p. In the latter case, the load intensity is considered to be zero at the inner punch edges.

The crack trajectories for a particular case of double shearing, obtained by using the two discretisation schemes (Figs. 7a and b), neglecting friction forces, and assuming the shearing load to be uniformly distributed, are compared in Fig. 8. The unstable crack path for a linearly distributed loading is also illustrated in Fig. 8. The punch edge crack trajectory is not significantly affected by the discretisation scheme. The influence of load distribution also appears to be inappreciable. The difference between the profiles based on the criteria of tangential stress and zero shear stress is included in Fig. 8.

At the tool specimen interfaces three values of friction coefficients (f) have been assumed in the range (0–0.6) for the determination of the crack trajectories. The results corresponding to a typical case of double shearing are shown in Fig. 9.

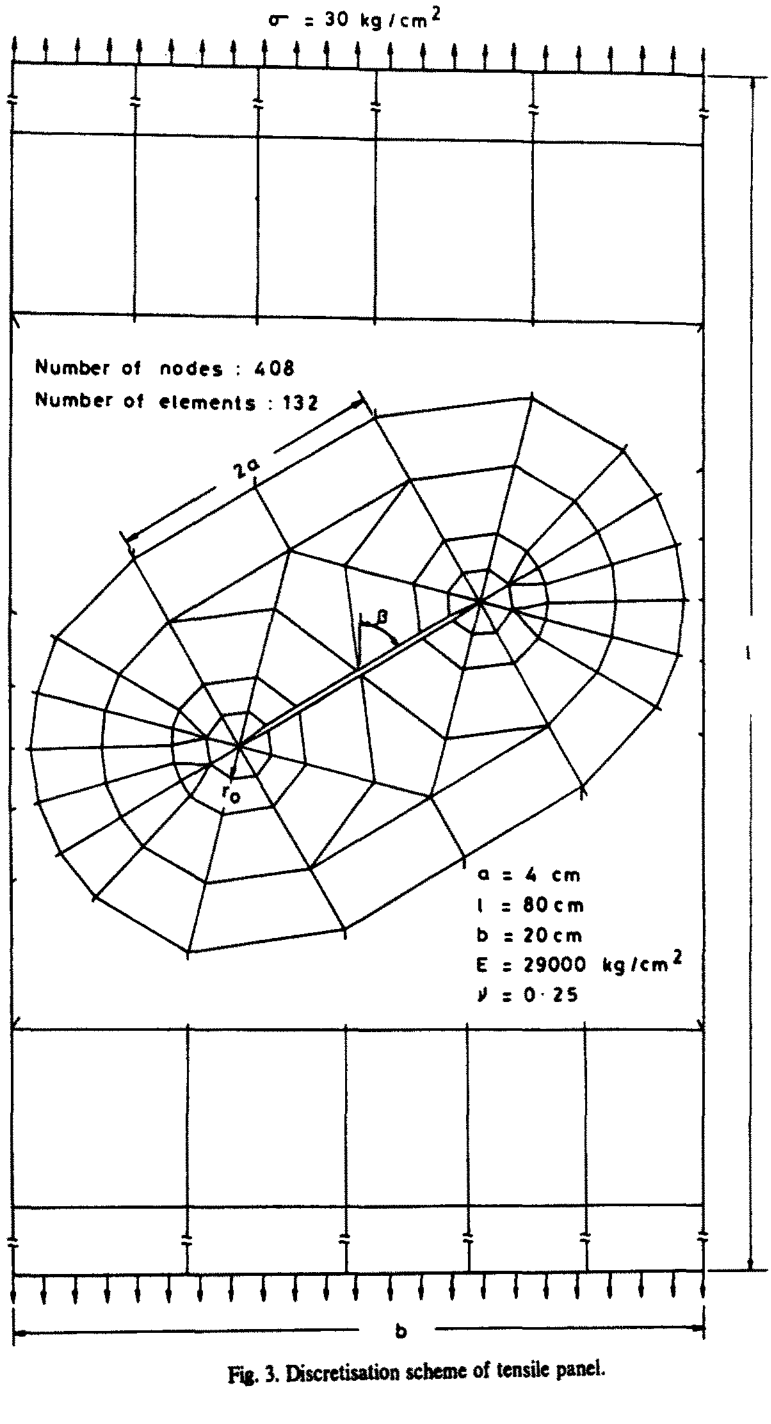


Fig. 3. Discretisation scheme of tensile panel.

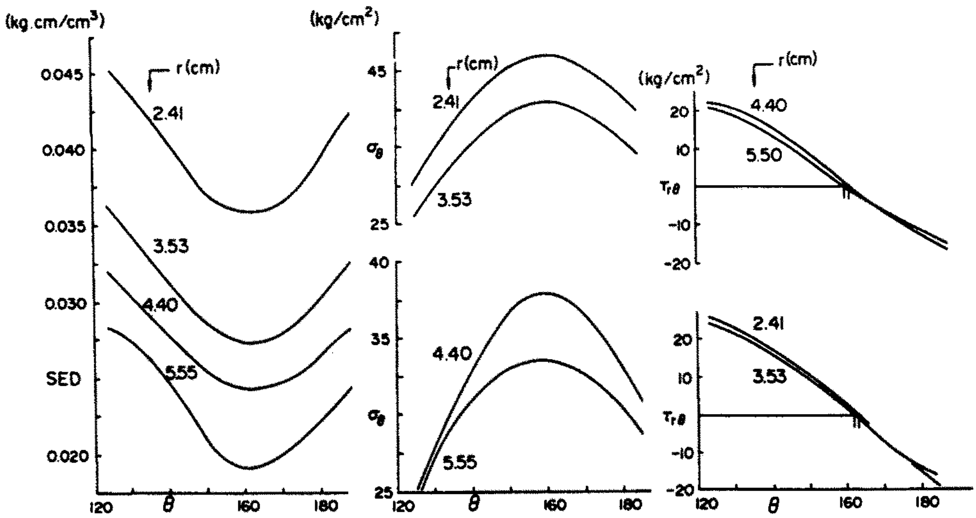


Fig. 4. Circumferential variation of SED, σ_θ and $\tau_{r\theta}$ in tensile panel ($\beta = 70°$).

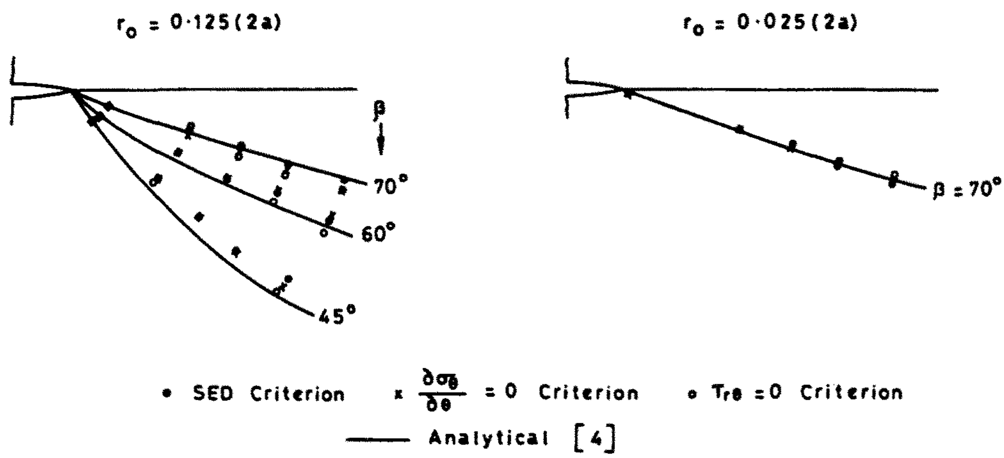


Fig. 5. Comparison of crack trajectories and the effect of crack tip element size.

The analysis of single shearing is almost similar, only a slight difference exists in the boundary condition at the right hand end (Fig. 7a).

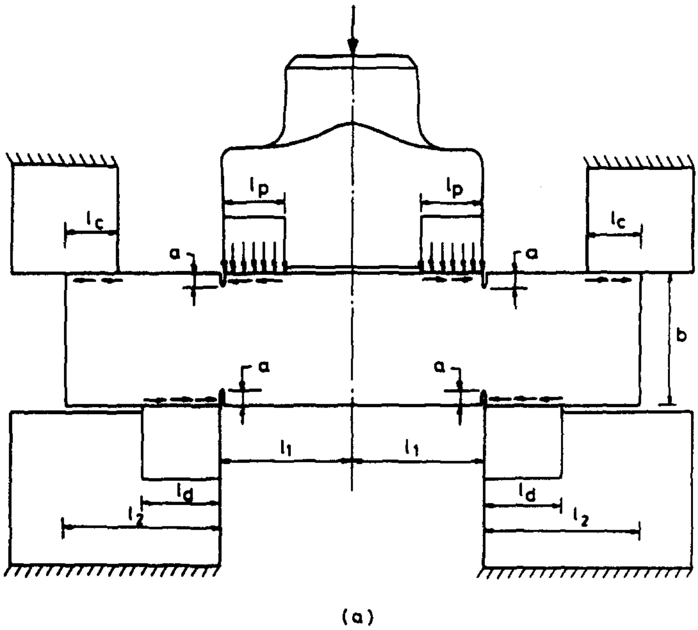
4. EXPERIMENTAL STUDIES

Experimental investigations on the unstable extension of edge cracks during shearing (by both the schemes) were carried out with specimens machined from 5–8 mm thick perspex (plexiglass) sheets to dimensions as shown in Fig. 10. A mild steel tool set with high speed steel die and punch inserts was used. The tool and the specimen geometry were made to have a general conformity with the configuration considered in the numerical studies. Both double and single shearing tests were conducted on a universal testing machine using the same tool set.

5. DISCUSSION

In the case of tensile panels (Fig. 5), the agreement between the numerical trajectories and the analytical results of Kipp and Sih[4] can be considered good in spite of the fact that the analytical results correspond to an infinite geometry. The change in the size r_0 of singularity elements (from 25% to 5% of the semi-crack length a) gives rise to a maximum difference of 2° in the angular location of a point on a particular circumference. This does not alter the crack trajectory significantly (Fig. 5).

Experimental observations indicate three possible modes of separation during double shearing, e.g. the separation may result from the unstable extension of the punch edge crack



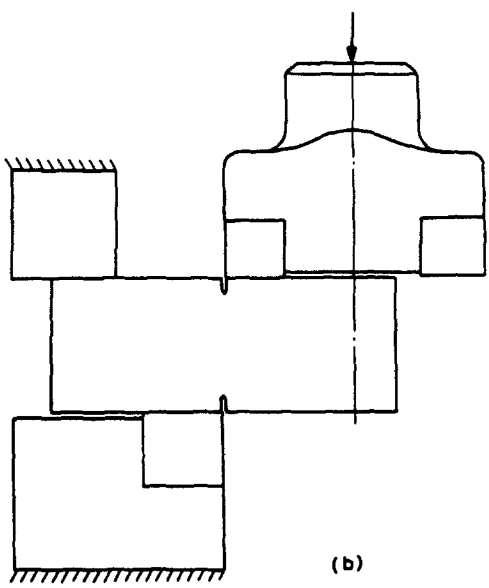


Fig. 6. Scheme of (a) double and (b) single shearing.

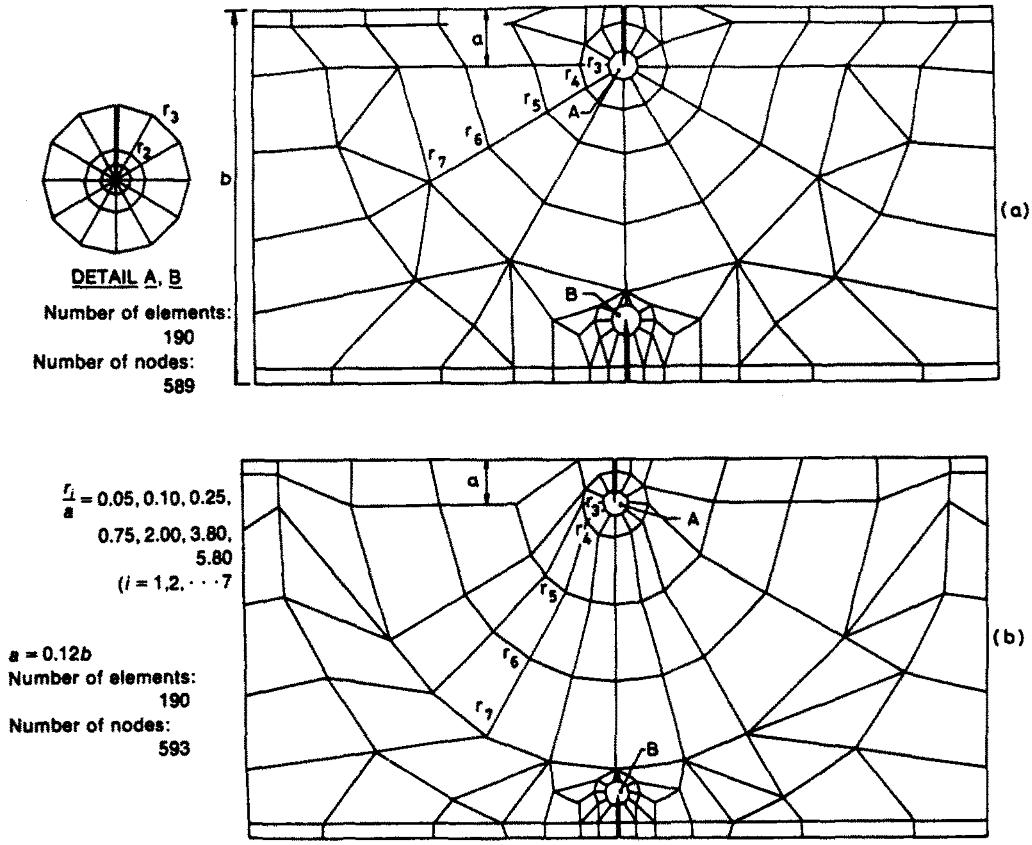


Fig. 7. Two discretisation schemes of shear specimen.

alone, the die edge crack alone, or by their simultaneous extensions [6]. Even out of these three modes, the spearation resulting from the extension of the punch edge crack alone is predominant. However, it is the only possible mode of separation in the case of single shearing. Therefore, the extension of the punch edge crack alone is chosen as the basis for the comparison of the three criteria. Some experimental observations on the punch edge crack extensions during double and single shearing are shown in Figs. 11 and 12 respectively.

For a realistic comparison of numerical and experimental results, it is necessary to have the exact correspondence in respect of load distribution, frictional characteristics at the interfaces, and geometry. Although the correspondence in respect of geometry can be ensured to a reasonable degree of accuracy the correspondence in respect of friction and load distribution can in no case be attained.

Numerical studies on the punch edge crack trajectory were carried out by changing the aspect ratio ($1_l/b$) (Fig. 6a) from 1 to 1.25 without altering any other dimension. Similar studies were carried out by changing the crack length ratio (a/b) from 0.12 to 0.16, punch width ratio ($1_p/b$) from 0.20 to 0.50, clamping length ratio ($1_c/b$) from 0.20 to 0.40, and the clearance between the punch and the die (on each side) from 0 to 10% of the section depth. The numerical results are not presented in this paper as they pertain more to a process study. However, it is sufficient here to mention that these parameters are insignificant. Thus the numerical punch edge crack profiles observed to be insensitive to all the parameters except the friction coefficient.

The experimental trajectories obtained from a number of specimens form a band that compares well with numerical results in the range 0.4–0.6 (Fig. 9). The dry friction coefficient for the tool-specimen combination [12] is around 0.5.

Similar observations pertain to the case of single shearing as can be seen from Fig. 13.

It is noted from Figs. 5, 9 and 13 that the criterion $\tau_{r\theta} = 0$ could be used for the prediction of the entire unstable crack path. Further the criterion appears to be more accurate than the tangential stress criterion, and the SED criterion is unsuitable, for the edge crack problems

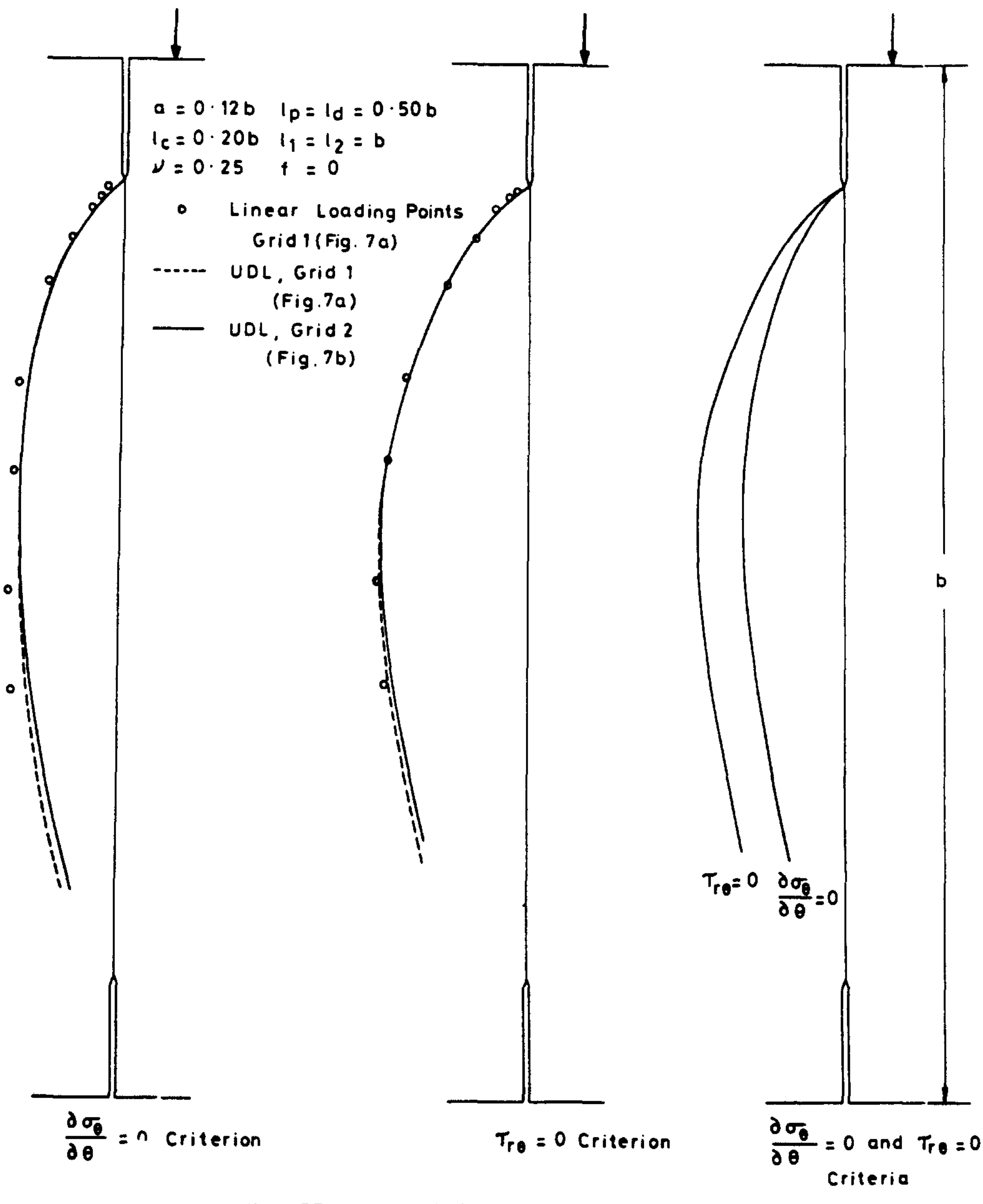


Fig. 8. Effects of load distribution and discretisation schemes.

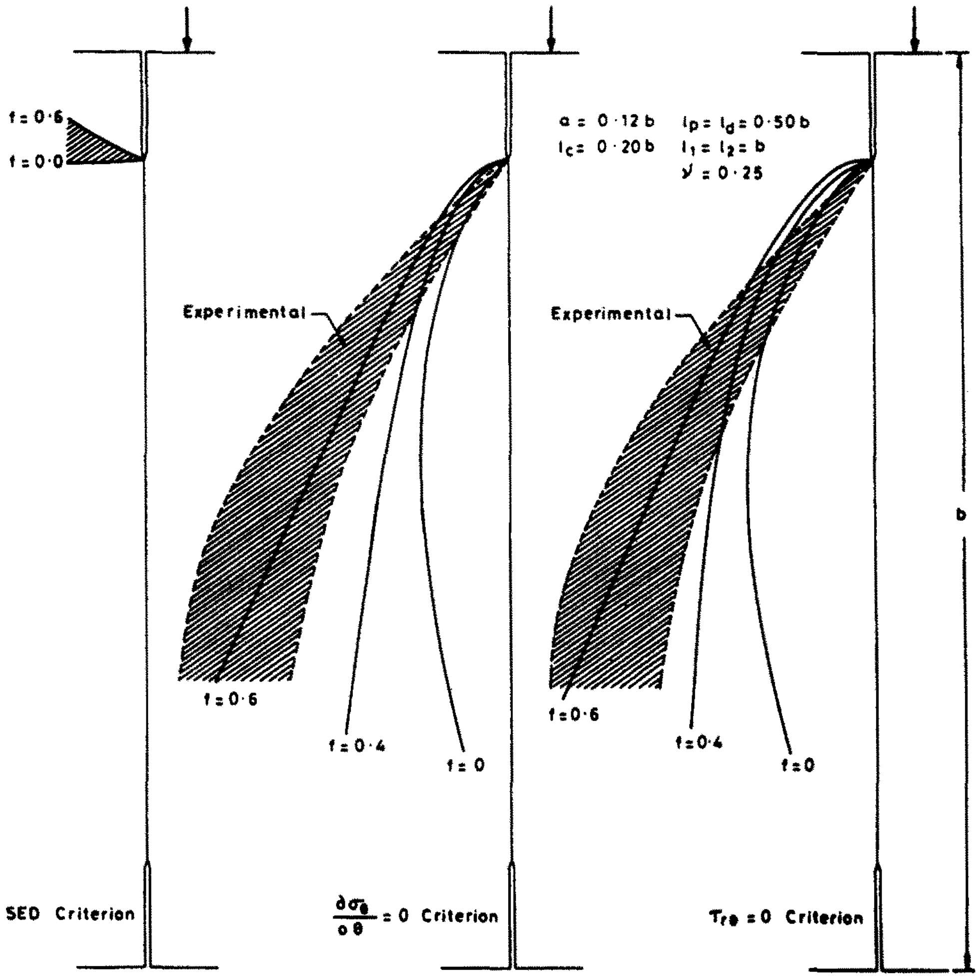


Fig. 9. Comparison of theoretical and experimental double sheared edge profiles.

considered. The zero shear stress criterion is computationally advantageous because $\tau_{r\theta}$ has almost a linear variation in the viscinity of $\tau_{r\theta} = 0$ (Fig. 4).

While applying the $\tau_{r\theta} = 0$ criterion at a particular radius outside the crack tip zone, in case these are two zeroes, the one closer to that of the adjacent radius has been taken for the determination of crack trajectories. This problem did not arise in the case of tensile panel, but this was quite pertinent (for $r/a \geq 0.625$ approximately) in the edge crack problems.

Although the difference between the trajectories based on the criteria of maximum tangential stress and zero shear stress is appreciable in the edge crack examples (Figs. 8, 9 and 13), it is not so in the case of internal crack in tensile panels (Fig. 5). The latter may be explained as follows:

If the stress field (ϕ) is dominated by one of the eigen functions ϕ_n (eqn3), then

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \simeq \frac{\partial^2\phi_n}{\partial r^2} = \left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right) r^{((n/2)-1)} f(n, \theta) \tag{5}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \simeq -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi_n}{\partial\theta}\right) = -\frac{n}{2} r^{((n/2)-1)} \frac{\partial f(n, \theta)}{\partial\theta} \tag{6}$$

where

$$f(n, \theta) = C_{2n-1}\left[\sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\sin\left(\frac{n}{2}+1\right)\theta\right] + C_{2n}\left[\cos\left(\frac{n}{2}-1\right)\theta - \cos\left(\frac{n}{2}+1\right)\theta\right]. \tag{7}$$

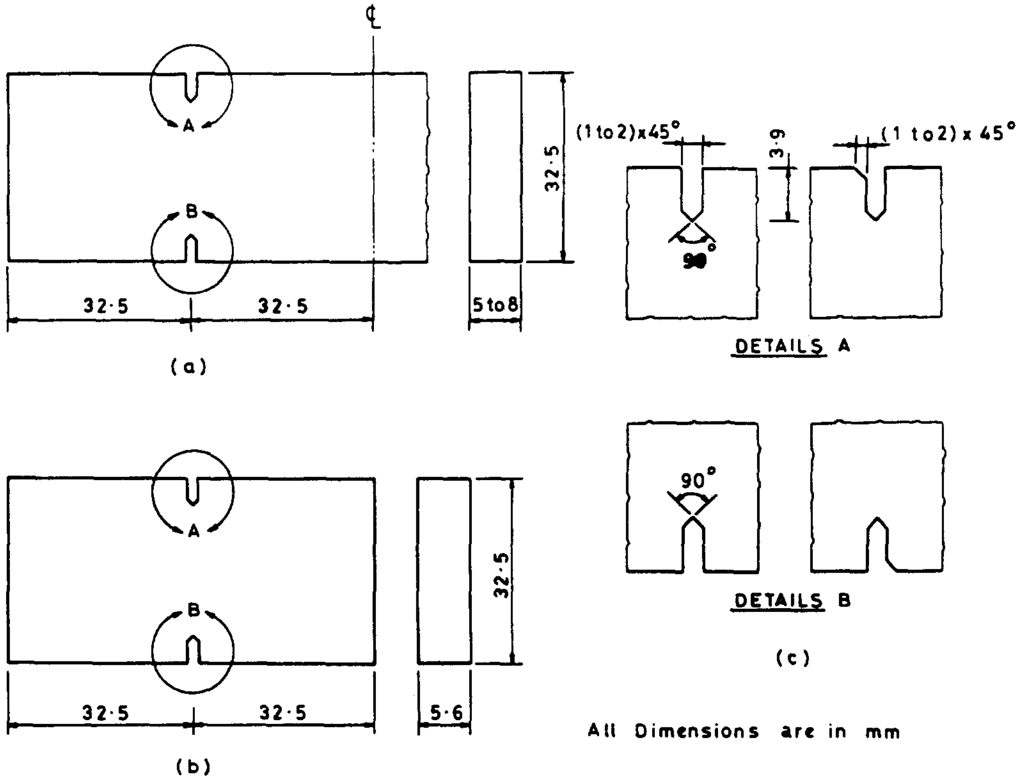


Fig. 10. Geometry of shear specimens.

Consequently, if $(\partial\sigma_\theta/\partial\theta) = 0$ at $\theta = \theta_c$ on the circumference of a circle, $\tau_{r\theta} = 0$ in its neighbourhood. This will lead to an inappreciable difference between the trajectories due to either criteria (see Fig. 5).

It is experimentally observed that the crack propagation takes place unstably and the surface of the sheared edge is perpendicular to the plane of the specimen, thereby validating the assumptions on the two-dimensional state of stress and the applicability of the LEFM principles.

Numerical studies indicate that the trajectory is insensitive to a slight error (upto 3° in θ) in the determination of a point on the trajectory. This may add to the reliability and validates the comparison of the crack extension criteria based on almost the entire crack trajectory. Further this results in considerable economy of analysis by dispensing with computer usage beyond the stage of calculation of Gauss point stresses and SEDs.

The observation that a running crack extends parallel to the direction of the maximum principal stress [13] stands in the way of application of the maximum tangential stress criterion at the crack tip. However, $\tau_{r\theta} = 0$ criterion does not appear to suffer from this limitation.

6. CONCLUSIONS

(1) The three criteria are useful for the prediction of the unstable crack path of an internal crack in tensile panels.

(2) The SED criterion appears to be unsuitable for the edge crack problem.

(3) The accuracy of the zero shear stress criterion appears to be higher than the maximum tangential stress criterion in the studied examples on edge crack extensions and the criterion is computationally advantageous.

(4) The experimental conditions favour the application of linear elastic fracture mechanics principles and justify the two-dimensional elastostatic stress analysis.

(5) The present type of analysis may be useful for understanding the mechanism of the shearing of brittle materials.

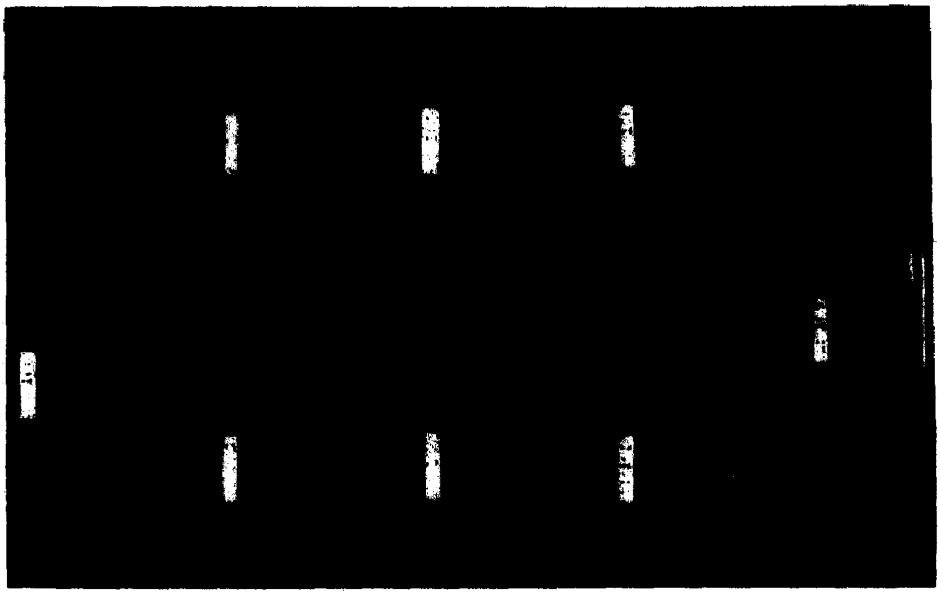


Fig. 12. Experimental edge crack profiles in single shearing.

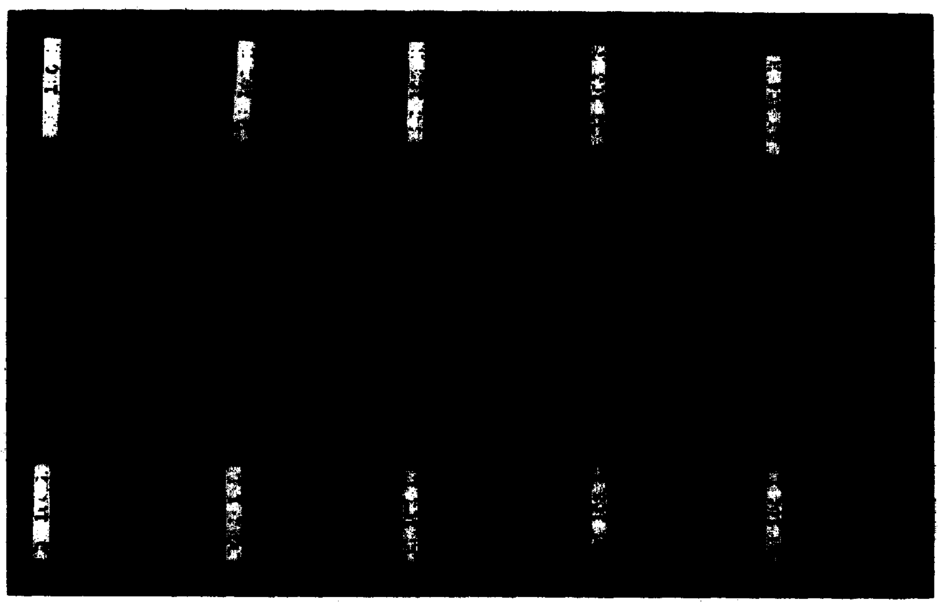


Fig. 11. Experimental edge crack profiles in double shearing.

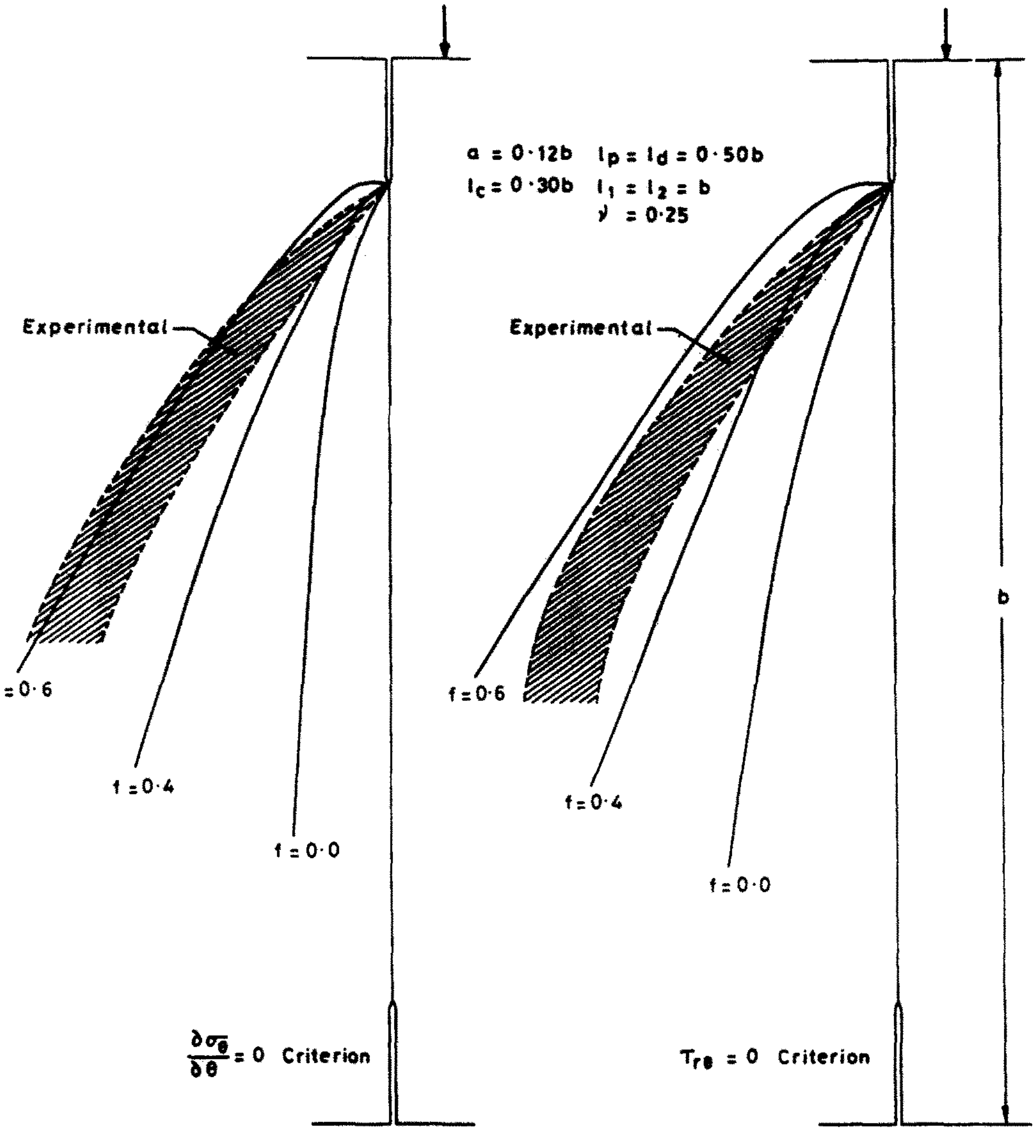


Fig. 13. Comparison of theoretical and experimental edge crack profiles in single shearing.

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